**z TRANSFORMS**

- z Transform Basics
- Transfer Functions
- Back to the Time Domain
- Transfer Function and Stability

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**z Transform Basics**

The definition of the **z transform** for a digital signal is:

\[ X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \]

- *z* is a **complex variable**
- The *z* transform of *x[n]*, called *X(z)* is said to lie in the *z domain*
- The *z* transform may not be defined for all value of *z*
- **Region of convergence (ROC)** – the values of *z* for which the *z* transform defines
z Transform Basics (cont.)

The z-transform is considered an operator that transforms a digital signal into its z domain:

\[ Z\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n} = X(z) \]

\(Z\{\cdot\}\) indicates a z transform is taken

The inverse z transform

\[ x[n] = Z^{-1}\{X(z)\} \]

See Examples 6.1 ~ 6.5

z Transform Basics (cont.)

For any signal with a finite number of samples, the ROC is \(z \neq 0\), i.e., all z except \(z = 0\)

See Table 6.1 for the z transform of basic signals
z Transform Basics (cont.)

**Time shifting property** – if $Z\{x[n]\} = X(z)$, then $Z\{x[n-1]\} = z^{-1}X(z)$

A factor $z^{-1}$ in the z domain corresponds to a one sample delay in the time domain

**General form:**

$Z\{x[n-k]\} = z^{-k}X(z)$

<table>
<thead>
<tr>
<th>Signal $x[n]$</th>
<th>$Z$ Transform $X(z)$</th>
<th>Region of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n]$</td>
<td>$1$</td>
<td>all $z$</td>
</tr>
<tr>
<td>$u[n]$</td>
<td>$\frac{z}{z - 1}$</td>
<td>$</td>
</tr>
<tr>
<td>$\beta^n[n]$</td>
<td>$\frac{z}{z - \beta}$</td>
<td>$</td>
</tr>
<tr>
<td>$na[n]$</td>
<td>$\frac{z}{(z - 1)^2}$</td>
<td>$</td>
</tr>
<tr>
<td>$\cos(n\Omega)n[n]$</td>
<td>$\frac{z^2 - z \cos \Omega}{z^2 - 2z \cos \Omega + 1}$</td>
<td>$</td>
</tr>
<tr>
<td>$\sin(n\Omega)n[n]$</td>
<td>$\frac{z \sin \Omega}{z^2 - 2z \cos \Omega + 1}$</td>
<td>$</td>
</tr>
<tr>
<td>$\beta^n\cos(n\Omega)n[n]$</td>
<td>$\frac{z^2 - \beta z \cos \Omega}{z^2 - 2\beta z \cos \Omega + \beta^2}$</td>
<td>$</td>
</tr>
<tr>
<td>$\beta^n\sin(n\Omega)n[n]$</td>
<td>$\frac{\beta z \sin \Omega}{z^2 - 2\beta z \cos \Omega + \beta^2}$</td>
<td>$</td>
</tr>
</tbody>
</table>
See Examples 6.6 & 6.7

Transfer Functions

Let $X(z)$ be the z transform of the input $x[n]$, $Y(z)$ be the z transform of the output $y[n]$, the transfer function $H(z)$ is defined as the ratio of $Y(z)$ to $X(z)$:

$$H(z) = \frac{Y(z)}{X(z)}$$

$y[n] = x[n] \ast h[n]$
Transfer Functions (cont.)

Given a general difference equation:

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]

or

\[ a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M] \]

Take z transform on both sides:

\[ a_0 Y(z) + a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z) \]

The transfer function is:

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \]

See Examples 6.8 ~ 6.12
Transfer Functions (cont.)

From the digital convolution

\[ y[n] = h[n] \ast x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n - k] \]

Take z transform on both sides:

\[ Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} X(z) = X(z) \sum_{k=-\infty}^{\infty} h[k] z^{-k} \]

\[ H(z) = \frac{Y(z)}{X(z)} = \sum_{k=-\infty}^{\infty} h[k] z^{-k} = Z\{h[n]\} \]

The transfer function is the z transform of the impulse response

Transfer Functions (cont.)

The impulse response is the inverse z transform of the transfer function:

\[ h[n] = Z^{-1}\{H(z)\} \]

See Example 6.13
Transfer Functions (cont.)

Finding filter outputs

- **Time domain**
  - Computed from the difference equation
  - Computed by using digital convolution

- **z domain**
  - Computed by using the transfer function

\[
H(z) = \frac{Y(z)}{X(z)} \Rightarrow Y(z) = H(z) X(z)
\]

Summary

- The z transform of the output, Y(z), is the product of the transfer function, H(z), in the z domain and the z transform of the input, X(z).

- The output of the filter can be taken by the inverse z transform of Y(z):

\[
y[n] = Z^{-1}\{Y(z)\}
\]

Convolution theorem – convolution in the time domain corresponds to multiplication in the z domain
Transfer Functions (cont.)

**Cascade combinations of filters**

![Diagram of cascade combination of filters]

$X(z) \rightarrow H_1(z)X(z) \rightarrow H_2(z) \rightarrow Y(z) = H_1(z)H_2(z)X(z)$

**Parallel combinations of filters**

![Diagram of parallel combination of filters]

$X(z) \rightarrow H_1(z)X(z) \rightarrow H_2(z)X(z) \rightarrow Y(z) = (H_1(z) + H_2(z))X(z)$

See Examples 6.14 ~ 6.16

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**Back to the Time Domain**

**Standard form of the transfer function**

- All exponents of $z$ in the $z$ transform be positive
- The coefficient of the highest power term in both numerator and the denominator be one

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 \left( 1 + \frac{b_1}{b_0} z^{-1} + \cdots + \frac{b_M}{b_0} z^{-M} \right)}{a_0 \left( 1 + \frac{a_1}{a_0} z^{-1} + \cdots + \frac{a_N}{a_0} z^{-N} \right)}$$

$$= \frac{b_0}{a_0} \left( z + \frac{b_1}{b_0} z^{-1} + \cdots + \frac{b_M}{b_0} z^{-M} \right) \left( z^N + \frac{a_1}{a_0} z^{N-1} + \cdots + \frac{a_N}{a_0} z^{-N} \right)$$
Back to the Time Domain (cont.)

A transfer function expressed in standard form is a rational function consisting of a numerator polynomial divided by a denominator polynomial.

- **Degree** of a polynomial – the highest power in a polynomial.
- **Proper rational function** – the degree of the numerator polynomial is less than or equal to the degree of the denominator polynomial.
- **Strictly proper rational function** – the degree of the numerator is less than the degree of the denominator.
- **Improper rational function** – the degree of the numerator is larger than the degree of the denominator.

See Examples 6.17 & 6.18.

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Back to the Time Domain (cont.)

- **Inverse z transform**
  - Inspection method
  - Power series expansion – long division
  - Partial fraction expansion

- Inspection method – using basic transforms listed in Table 6.1

See Examples 6.19 ~ 6.23.
Back to the Time Domain (cont.)

Power series expansion – using long division

If the z-transform is given as a power series of the form:

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ = \cdots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^1 + x[2]z^2 + \cdots \]

The sequence value \( x[n] \) are the coefficients of \( z^n \)

See Examples 6.24 & 6.25

Partial fraction expansion – useful for a strictly proper rational function in standard form

An example: \( x[n] = u[n-1] \), \( h[n] = (-0.25)^n u[n] \), \( y[n] \)?

\[ X(z) = Z\{x[n]\} = Z\{u[n-1]\} = z^{-1} \frac{z}{z-1} = \frac{1}{z-1} \quad \therefore Z\{u[n]\} = \frac{z}{z-1} \]

\[ H(z) = Z\{h[n]\} = \frac{z}{z + 0.25} \quad \therefore Z\{\beta^n u[n]\} = \frac{z}{z - \beta} \]

\[ Y(z) = H(z)X(z) = \frac{z}{(z + 0.25)(z - 1)} \]
Back to the Time Domain (cont.)

Step 4: represent $Y(z)$ by partial fraction expansion

$$Y(z) = \frac{z}{(z + 0.25)(z - 1)} = \frac{A}{z + 0.25} + \frac{B}{z - 1}$$

**Cover-up method** for finding the coefficients $A$ and $B$

- $A$ – multiply both side by $(z+0.25)$ and then set $z = -0.25$
  $$z + 0.25)Y(z) = \frac{z}{z - 1} = A + \frac{B(z + 0.25)}{z - 1}$$
  Set $z = -0.25 : A = \frac{-0.25}{-0.25 - 1} = 0.2$

- $B$ – multiply both side by $(z-1)$ and then set $z = 1$
  $$\frac{z}{z + 0.25} = \frac{A(z - 1)}{z + 0.25} + B$$
  Set $z = 1 : B = \frac{1}{1 + 0.25} = 0.8$

Step 5: take the inverse z transform of $Y(z)$

$$y[n] = 0.2(-0.25)^{n-1}u[n - 1] + 0.8u[n - 1]$$

See Examples 6.26 ~ 6.29
Transfer Functions and Stability

- **Poles** – the values of $z$ that make the denominator of a transfer function zero
- **Zeros** – the values of $z$ that make the numerator of a transfer function zero

Given a general form of a transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- $M$ zeros: $z_1, z_2, \ldots, z_M$
- $N$ poles: $p_1, p_2, \ldots, p_N$

Transfer Functions and Stability (cont.)

- The transfer function can be written as

$$H(z) = \frac{K(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}, \quad K = \frac{b_0}{a_0}$$

- $z_i$: zeros of the filter
- $p_j$: poles of the filter
- $K$: gain of the filter

- **z-plane** – the complex plane on which the poles and zeros of the transfer function are plotted

- See Examples 6.30 ~ 6.32
Transfer Functions and Stability (cont.)

- **Stable system** – every bounded input (finite in size) produces bounded output
  - \(| x[n] | < B_1 \Rightarrow | y[n] | < B_2 , \forall n\)

- If a filter is unstable, output grows without bound
- The output from an unstable filter can change dramatically even when the input changes by only the smallest amount
- All useful filters are stable and an important aspect of filter design is to **guarantee stability**

If the input \(x[n]\) is bounded, i.e., \(| x[n] | < B, \forall n\) then

\[
| y[n] | = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|
\]

- If the impulse response is **absolutely summable**, i.e., if \(\sum_{k=-\infty}^{\infty} |h[k]| < \infty\)
- The system is stable
Transfer Functions and Stability (cont.)

Fourier transform: \[ H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \]

z transform: \[ H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \]

If \( z = e^{j\omega} \) with \( \omega \) real (i.e., \( |z|=1 \)), the z transform of \( h[n] \) corresponds to the discrete-time Fourier transform of \( h[n] \)

Unit circle – a circle with radius one centered at the origin of the z plane

If the Fourier transform of \( h[n] \) converges

- The ROC of \( H(z) \) must include the unit circle

For a causal system with rational transfer function, the ROC is outside the outermost pole

- If the ROC includes the unit circle, \( |z|=1 \), all of the poles must be inside the unit circle

A causal digital filter with rational transfer function \( H(z) \) is stable if and only if all of the poles of \( H(z) \) lie inside the unit circle – i.e., they must all have magnitude smaller than 1
Transfer Functions and Stability (cont.)

**Summary**
- **Stable** – all the poles of the filter are inside the unit circle
- **Marginally stable** – with some poles on the unit circle
- **Unstable** – with some poles outside the unit circle
- **The ROC for a stable transfer function must include the unit circle**

See Examples 6.33 & 6.34
Transfer Functions and Stability (cont.)

First order system

- A simple first order system is: \[ H(z) = \frac{1}{1 + \alpha z^{-1}} = \frac{z}{z + \alpha} \]
- Has just one pole at \( z = -\alpha \)
- Requirement for stability: \( |\alpha| < 1 \)
- Impulse response: \( h[n] = (-\alpha)^n u[n] \)
  - When \( |\alpha| > 1 \), \( h[n] \) grows without bound as \( n \) increases
  - When \( |\alpha| < 1 \), \( h[n] \) settles down to zero
- Difference equation: \[ y[n] + \alpha y[n-1] = x[n] \]
  - The step response settles to a constant value \( y_{ss} \) in steady state:
    \[ y_{ss} + \alpha y_{ss} = 1 \implies y_{ss} = \frac{1}{1 + \alpha} \]

Transfer Functions and Stability (cont.)

(a) Pole-Zero Plots

\( x > 0 \)

\( x < 0 \)

DSP-G 6.31

DSP-G 6.32
Transfer Functions and Stability (cont.)

(b) Impulse Responses

α > 0

α < 0

(c) Step Responses

See Examples 6.35 & 6.36
Transfer Functions and Stability (cont.)

Second order system

- The transfer function of a simple second order system is:
  \[ H(z) = \frac{1}{1 + \alpha z^{-1} + \beta z^{-2}} = \frac{z^2}{z^2 + \alpha z + \beta} = \frac{z^2}{(z - p_1)(z - p_2)} \]
- \( p_1 \) and \( p_2 \) are the two poles of the transfer function
- Has two zeros at \( z = 0 \)
- Requirement for stability: \( |p_1| < 1 \) and \( |p_2| < 1 \)

Transfer Functions and Stability (cont.)

Second order system

- Difference equation: \( y[n] + \alpha y[n-1] + \beta y[n-2] = x[n] \)
  - The steady state value \( y_{ss} \) can be predicted by:
    \[ y_{ss} + \alpha y_{ss} + \beta y_{ss} = 1 \Rightarrow y_{ss} = \frac{1}{1 + \alpha + \beta} \]

See Examples 6.37 ~ 6.41
Transfer Functions and Stability (cont.)

(a) Poles: \(-0.1, -0.1\)
Magnitude of poles: 0.1, 0.1

(b) Poles: \(-0.3 \pm j0.2\)
Magnitude of poles: 0.3606, 0.3606
Transfer Functions and Stability (cont.)

(c) Poles: 0.6, 0.6
Magnitude of poles: 0.6, 0.6

(d) Poles: 0.5 ± j0.5
Magnitude of poles: 0.7071, 0.7071

DSP-G 6.39

Transfer Functions and Stability (cont.)

DSP-G 6.40
Transfer Functions and Stability (cont.)

(e) Poles: 0.35, 0.8
Magnitude of poles: 0.35, 0.8

Transfer Functions and Stability (cont.)

(f) Poles: −0.85 ± j0.2
Magnitude of poles: 0.8732, 0.8732
Transfer Functions and Stability (cont.)

(g) Poles: -0.9, -0.9
Magnitude of poles: 0.9, 0.9

Transfer Functions and Stability (cont.)

(h) Poles: 0.8 ± 0.55
Magnitude of poles: 0.9706, 0.9708
Transfer Functions and Stability (cont.)

The magnitudes of the poles have a large impact on the time it takes the system to settle to its final value:

- The closer a pole is to the edge of the unit circle, the longer it takes for the output to settle.
- The closer a pole is to the center of the unit circle, the faster the output settles.

The magnitudes of the zeros can modify the behavior of the output dramatically:

- The closer the zeros are to the poles, the greater their effect on system behavior.

(a) Zeros: 0, 0
Transfer Functions and Stability (cont.)

(b) Zeros: 0, 0.3

Transfer Functions and Stability (cont.)

(c) Zeros: 0, 0.8
Transfer Functions and Stability (cont.)

(d) Zeros: $0.8 - j0.4$